# On Character Amenability of Banach Algebras

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**Abstract**

Associated to a nonzero homomorphism $\varphi$ of a Banach algebra $A$, we regard special functionals, say $m_\varphi$, on certain subspaces of $A^*$ which provide equivalent statements to the existence of a bounded right approximate identity in the corresponding maximal ideal in $A$. For instance, applying a fixed point theorem yields an equivalent statement to the existence of a $m_\varphi$ on $A^*$; and, in addition we expatiate the case that if a functional $m_\varphi$ is unique, then $m_\varphi$ belongs to the topological centre of the bidual algebra $A^{**}$. An example of a function algebra, surprisingly, contradicts a conjecture that a Banach algebra $A$ is amenable if $A$ is $\varphi$-amenable in every character $\varphi$ and if functionals $m_\varphi$ associated to the characters $\varphi$ are uniformly bounded. Aforementioned are also elaborated on the direct sum of two given Banach algebras.

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## 1 Introduction

A. T. M. Lau introduced and studied the left amenability of certain Banach algebras [17]. In the special cases he pointed out that for a locally compact group $G$, the left amenability of $L^1(G)$, the group algebra, and the left amenability of $M(G)$, the measure algebra, are equivalent to the amenability of $G$. The latter, due to B. Johnson [15], is also equivalent to the amenability of $L^1(G)$ as well as the existence of a bounded right approximate identity in the maximal ideal of $L^1(G)$, consisting of functions with vanishing integral. Hence, the left amenability of $L^1(G)$ (with respect to the identity character) is equivalent to the amenability of $L^1(G)$. We may refer the reader e.g. to [7, 11, 21] and [23] for extensive treatments of various notions of amenability. One may now pose a natural question of whether the above statements remain valid for a Banach algebra with a nonzero homomorphism on it.

In [1] we initiated and studied the notion of (nontrivial) character (left) amenability for a commutative hypergroup algebra, and, among other things, we showed that a hypergroup algebra is character (left) amenable if and only if its maximal ideal associated to the character has a bounded approximate identity. Moreover, with a large family of examples of hypergroups we revealed that this kind of amenability of hypergroup algebras depends heavily on the asymptotic behavior of both Haar measures of hypergroups and characters. In particular, the algebra may in general fail to be even weakly amenable [1, 24]. So, we may thence, in general, infer a negative answer to the anterior question.

In this work we aim to develop a notion of character amenability for Banach algebras, which, in particular, we generalize major results in [17]. The article is organized as follows:

Section 2 briefly presents pertinent notions and notations on Banach algebras, whereas Sections 3, 4 and 5 contain main results and examples. Let $A$ denote a Banach algebra and $Hom(A,\mathbb{C})$ the set of all nonzero homomorphisms (characters) from $A$ into $\mathbb{C}$. Suppose $\varphi \in Hom(A,\mathbb{C})$ and $I_\varphi(A)$ designates the maximal ideal in $A$ determined by $\varphi$. Theorems 3.1, 3.3, and 3.4 provide equivalent statements to the $\varphi$-amenability of $A$. Additionally, Theorem 3.4 also utters that $A$ is $\varphi$-amenable with a bounded right approximate identity if and only if $I_\varphi(A)$ has a bounded right approximate identity. If $A$ is commutative, regular, and semisimple, and $\{\varphi\}$ is a spectral set, then by Proposition 3.6 every $\varphi$-derivation on $A$ is zero. The $\varphi$-left amenability of certain subspaces of $A^*$ with a unique $\varphi$-mean is elaborated in Theorem 3.10 and Remark 3.2.

In Section 4 we consider the Banach algebra $A\oplus_\phi B$ of direct sum of given Banach algebras $A$ and $B$, where $\phi \in Hom(B,\mathbb{C})$. In this section, we first show that $(A\oplus_\phi B)^{**} \cong A^{**}\oplus_\phi B^{**}$ (Proposition 4.3), and then Theorem 4.4 indicates the isomorphism $Z\left((A\oplus_\phi B)^{**}\right) \cong Z(A^{**})\oplus_\phi Z(B^{**})$ of topological centres. Proposition 4.5 determines $Hom(A\oplus_\phi B,\mathbb{C}) \cong Hom(A,\mathbb{C})\times\{\phi\}$, and Theorem 4.7 affirms that $A\oplus_\phi B$ is $\varphi\times\phi$-amenable if and only if $A$ is $\varphi$-amenable.

Section 5 concludes the paper with examples of Banach function algebras whose $\varphi$- amenability depends on characters $\varphi$.

## 2 Preliminaries

For a linear space $X$ and a function $f$ on $X$, the value of $f$ at $x\in X$ is denoted by $f(x)$, $\langle f,x\rangle$ or $\langle x,f\rangle$. The usual dual and bidual spaces of $X$ are specified by $X^*$ and $X^{**}$, respectively; and, $\pi$ designates the canonical embedding of $X$ into $X^{**}$.

Let $A$ be a Banach algebra and $X$ a subspace of $A^*$. For $a\in A$ and $f\in X$ define $a\cdot f$ and $f\cdot a$ of $A^*$ by

$$\langle f\cdot a,b\rangle = \langle f,ab\rangle, \quad \langle a\cdot f,b\rangle = \langle f,ba\rangle, \qquad (b\in A).$$

$X$ is called $A$-left (resp. right) invariant if $X \cdot A \subseteq X$ (resp. $A \cdot X \subseteq X$); it is called $A$-invariant if it is $A$-left and right invariant. For $m \in X^*$ and $f \in X$ define $m \cdot f \in A^*$ by

$$\langle m \cdot f, a \rangle = \langle m, f \cdot a \rangle, \qquad (a \in A).$$

When $X = A^*$, $X$ is a Banach $A$-bimodule, and $A^{**}$ with the following Arens product is a Banach algebra [5]:

$$\langle n \cdot m, f \rangle = \langle n, m \cdot f \rangle, \qquad (m, n \in A^{**}, f \in A^*).$$

Let $Hom(A, \mathbb{C})$ denote the set of all nonzero homomorphisms from $A$ into $\mathbb{C}$. For every $\varphi \in Hom(A, \mathbb{C})$, we have $\pi(\varphi) \in Hom(A^{**}, \mathbb{C})$ [6], and $I_\varphi(A) := \{a \in A : \varphi(a) = 0\}$ is a maximal ideal in $A$. Further, if $A$ is commutative, then $Hom(A, \mathbb{C})$ is called the character space of $A$ and will be denoted by $\Phi(A)$. Observe that $\Phi(A)$ is a locally compact Hausdorff space with the $w^*$-topology, and every maximal ideal of $A$ is given by $I_\varphi(A)$, for some $\varphi \in \Phi(A)$ [5]. A net $\{e_i\}_i$ in $A$ ($\|e_i\| < M$ for some $M > 0$) is called a bounded right (resp. left) approximate identity if $\|ae_i - a\| \to 0$ (resp. $\|e_i a - a\| \to 0$ ) as $i \to \infty$, for every $a \in A$. It is called a bounded approximate identity, if it is a bounded left and right approximate identity.

**Definition 2.1.** Let $A$ be a Banach algebra and $X$ a linear, closed, $A$-left invariant subspace of $A^*$ such that $Hom(A, \mathbb{C}) \subseteq X$. $X$ is called $\varphi$-left amenable if there exists a $m_\varphi \in X^*$ with the following properties:

(i) $m_\varphi(\varphi) = 1$

(ii) $m_\varphi(f \cdot a) = \varphi(a) m_\varphi(f)$, for every $f \in X$ and $a \in A$.

Plainly the dual space $A^*$ is linear, closed, and $A$-invariant. A Banach algebra $A$ is called $\varphi$-amenable if $A^*$ is $\varphi$-left amenable.

Let $B(A^*)$ denote the space of bounded linear operators of $A^*$ into $A^*$. A net of operators $\{T_i\}$ in $B(A^*)$ is said to be converges to $T$ in the $w^*$-operator topology, denoted by $w^*ot$ , if and only if for every $a \in A$, $T_i a \to Ta$ in the $w^*$-topology on $A^*$, as $i \to \infty$. As shown in [16], the unit ball of $B(A^*)$ is compact in the $w^*ot$.

# 3 $\varphi$-Left Amenability of Subspaces of $A^*$

**Theorem 3.1.** Let $A$ be a Banach algebra and $X$ a linear, closed, $A$-left invariant subspace of $A^*$ such that $Hom(A, \mathbb{C}) \subseteq X$. Suppose $P : A^* \to X$ is a continuous linear map such that $P(Hom(A, \mathbb{C})) \subseteq Hom(A, \mathbb{C})$ and $P(f \cdot a) = P(f) \cdot a$, for every $f \in A^*$ and $a \in A$. Then $X$ is $\varphi$-left amenable if and only if $A$ is $\varphi$-amenable.

*Proof.* Let $X \subset A^*$ be $\varphi$-left amenable. Then there exists a $m_\varphi \in X^*$ such that $m_\varphi(\varphi) = 1$ and $m_\varphi(f \cdot a) = \varphi(a)m_\varphi(f)$, for every $f \in X$ and $a \in A$. Since $P(\varphi) \cdot a = \varphi(a)P(\varphi)$ for every $a \in A$, we have $P(\varphi) = \varphi$. The continuity of $P$ implies that the functional $m'_\varphi : A^* \to \mathbb{C}$, defined by $m'_\varphi := m_\varphi \circ P$, belongs to $A^{**}$, and $m'_\varphi(\varphi) = m_\varphi(P\varphi) = m_\varphi(\varphi) = 1$. Moreover, for every $f \in A^*$ and $a \in A$, we may have

$$\begin{aligned} m'_\varphi(f \cdot a) &= m_\varphi \circ P(f \cdot a) \\ &= m_\varphi(P(f \cdot a)) \\ &= m_\varphi(P(f) \cdot a) \\ &= \varphi(a)m_\varphi(Pf) \\ &= \varphi(a)m_\varphi \circ P(f). \end{aligned}$$

Hence, $m'_\varphi$ is a $\varphi$-mean on $A^*$. The converse of the theorem is trivial.

□

**3.2. Examples and Remarks:**

(i) Let $A$ be a Banach algebra with a bounded approximate identity and $X = AA^*$. We observe that $X$ is a (proper) closed submodule of $A^*$ [13], and $Hom(A, \mathbb{C}) \subset AA^*$. For a fixed $b \in A$ the map $P_b : A^* \to X$ defined by $P_b(f) = b \cdot f$ is a continuous linear map and $P_b(f \cdot a) = P_b(f) \cdot a$, for every $a \in A$. Then by the previous theorem $X$ is $\varphi$-left amenable if and only if $A$ is $\varphi$-amenable.
We remark that two $\varphi$-means $m_\varphi$ and $m'_\varphi$ on $A^*$ are equal if and only if they are equal on $A^*A$, since

$$\varphi(a)m_\varphi(f) = m_\varphi(f.a) = m'_\varphi(f.a) = \varphi(a)m'_\varphi(f), \qquad \text{for every } f \in A^* \text{ and } a \in A.$$

(ii) Let $G$ be a locally compact group and $A = L^1(G)$ the group algebra. Then $A^* \cong L^\infty(G)$ and $L^1(G) * L^\infty(G) \cong UC_r(G)$, the algebra of bounded uniformly right continuous functions on $G$. By part (i) and the previous theorem, $UC_r(G)$ is 1-amenable if and only if $G$ is amenable which is already known e.g. in [21]. It is worth noting that, $AP(G)$ [$WAP(G)$], the Banach algebra of [weakly] almost periodic functions on $G$, is norm closed two sided translation invariant subspace of $L^\infty(G)$ and is always 1-amenable even if $G$ is not amenable [21, p.88].

(iii) Notice that in the above theorem, the subspace $X$ of $A$ need not to be an algebra. For example, if $K$ is a locally compact hypergroup, then $L^1(K) * L^\infty(K) \cong UC_r(K)$ may, in general, fail to be an algebra . However, $UC_r(K)$ is 1-amenable if and only if $K$ is amenable [24].

And, by part (i) and the preceding theorem, $UC_r(K)$ is $\varphi$-amenable if and only if $L^1(K)$ is $\varphi$-amenable, which is already known in [2] for commutative hypergroups.

**Theorem 3.3.** Let $A$ be a commutative Banach algebra and $\varphi \in \Phi(A)$. Then $A$ is $\varphi$-amenable if and only if for every $f \in A^*$, $\lambda\varphi \in \overline{K(f)}^{w^*}$ for some $\lambda \in \mathbb{C}$, where $K(f) = \{f \cdot a;\ \ a \in A\}$ is considered with the $w^*$-topology.

*Proof.* Let $A$ be a $\varphi$-amenable Banach algebra and $m_\varphi \in A^{**}$, a $\varphi$-mean on $A^*$. By Goldstein's theorem [10], there exists $\{m_{(\varphi,i)}\}_i$ a net in $A$ such that $\pi(m_{(\varphi,i)}) \xrightarrow{w^*} m_\varphi$, and hence $\pi(m_{(\varphi,i)}) \cdot a \xrightarrow{w^*} m_\varphi \cdot a$, for every $a \in A$, as $i \to \infty$. Thus

$$\lim_{i\to\infty} \langle \pi(m_{(\varphi,i)}) \cdot a, f\rangle = \langle m_\varphi, f \cdot a\rangle = \varphi(a)\langle m_\varphi, f\rangle, \quad \text{for every } f \in A^*.$$

By letting $\langle m_\varphi, f\rangle = \lambda$, we find $\lambda\varphi \in \overline{K(f)}^{w^*}$.

In order to prove the converse of the theorem, let $S = \{T_m :\ \ T_m : f \mapsto m \cdot f, f \in A^*, m \in B'\}$, where $B'$ denotes the bidual space of the unit ball $B$ in $A$. Since the unit Ball in $B(A^*)$ is compact [16], the set $\overline{S}$, the $(w^*ot)$ closure of $S$ in $B(A^*)$, is compact. Let $f \in A^*$ and define $S_f := \{T_m f :\ m \in B'\}$ and $[A^*]_{(S,\varphi)} := \{f \in A^* :\ T_m f = m(\varphi) f,\ \text{for every } m \in B'\}$. We observe that $\overline{S_f}^{w^*} \cap [A^*]_{(S,\varphi)}$ is nonempty for every $f \in A^*$; in fact, since $\lambda\varphi \in \overline{\{T_{\pi(a)} f,\ a \in B\}}^{w^*}$, there exists $\{a_i\}$ a net in $B$ such that $T_{\pi(a_i)} f \xrightarrow{w^*} \lambda\varphi$, as $i \to \infty$. Since $B'$ is $w^*$-compact (Banach Alaoglu's theorem), we may let $m_{(f,\varphi)}$ be a $w^*$-accumulation point of $\{\pi(a_i)\}_i$ in $B'$. Then

$$\begin{aligned}\langle m_{(f,\varphi)} \cdot f, b\rangle &= \langle m_{(f,\varphi)}, f \cdot b\rangle \\ &= \lim_{i\to\infty} \langle \pi(a_i), f \cdot b\rangle = \lambda\langle \varphi, b\rangle, \quad \text{for every } b \in A,\end{aligned}$$

i.e., $T_{m_{(f,\varphi)}} f = \lambda\varphi$. Thus

$$T_m T_{m_{(f,\varphi)}}(f) = T_m\left(T_{m_{(f,\varphi)}} f\right) = \lambda T_m \varphi = \lambda m \cdot \varphi = \lambda m(\varphi)\varphi = m(\varphi) T_{m_{(f,\varphi)}} f, \text{ for every } m \in B',$$

which implies that $T_{m_{(f,\varphi)}} \in [A^*]_{(S,\varphi)}$. For any $f \in A^*$, let $Z(f) = \left\{T_m : T_m f \in [A^*]_{(S,\varphi)}\right\}$. The subsets $Z(f)$ of $S$ are closed and have the finite intersection property. To verify this, suppose that $\{f_1, f_2, \dots f_n\}$ is any finite subset of $A^*$ and $T_m \in \bigcap_{i=1}^{n-1} Z(f_i)$, then $T_m f_i \in [A^*]_{(S,\varphi)}$. Since $\overline{S_{T_m f_n}}^{w^*} \cap [A^*]_{(S,\varphi)} \neq \emptyset$, there exists a $m' \in B'$ such that $T_{m'}(T_m f_n) \in [A^*]_{(S,\varphi)}$, accordingly $T_{m' \cdot m} \in \bigcap_{i=1}^{n} Z(f_i)$. Because of $T_m f_i \in [A^*]_{(S,\varphi)}, 1 \le i \le n-1$, we have

$$\begin{aligned}T_n T_{m' \cdot m} f_i &= T_n(T_{m'} T_m f_i) \\ &= m'(\varphi) T_n T_m f_i \\ &= m'(\varphi) n(\varphi) T_m f_i \\ &= n(\varphi) T'_m T_m f_i \\ &= n(\varphi) T_{m' \cdot m} f_i, \qquad \text{for every } n \in B', 1 \le i \le n-1.\end{aligned}$$

The compactness of $\overline{S}$ yields $\bigcap_{f\in A^*} Z(f) \neq \emptyset$; so, let $P$ be an element of this intersection. There exists $\{T_{m_i}\} \subseteq \bigcap_{f\in A^*} Z(f)$ a net of operators such that $T_{m_i} \xrightarrow{w^*op} P$, as $i \to \infty$. Then

$$\langle P(T_a f), b\rangle = \lim_{i\to\infty}\langle T_{m_i}(T_a f), b\rangle = \lim_{i\to\infty}\langle T_a(T_{m_i} f), b\rangle = \varphi(a)\lim_{i\to\infty}\langle T_{m_i} f, b\rangle = \varphi(a)\langle P(f), b\rangle,$$

for every $f \in A^*$, $a \in B$ and $b \in A$. Now let $m_\varphi \in A^{**}$ such that $m_\varphi(P(\varphi)) = 1$, and define $m : A^* \longrightarrow \mathbb{C}$ by $m(f) = m_\varphi(P(f))$. Because of $m(\varphi) = m_\varphi(P(\varphi)) = 1$, and

$$\begin{aligned} m(f\cdot a) &= m(T_a f) \\ &= m_\varphi(P(T_a f)) \\ &= \varphi(a) m_\varphi(P(f)) \\ &= \varphi(a) m(f), \quad \text{for every } a \in B \text{ and } f \in A^*, \end{aligned}$$

$m$ is a $\varphi$-mean on $A^*$. □

Let $X$ be a Banach $A$-bimodule and $\varphi \in Hom(A,\mathbb{C})$. Then $X^*$ in a canonical way is a Banach $A$-bimodule. A Banach $A$-bimodule $X$ is called a Banach $\varphi$-left (resp. right) $A$-bimodule if the left (resp. right) module multiplication is $a\cdot x = \varphi(a)x$ (resp. $x\cdot a = \varphi(a)x$), for all $a \in A$ and $x \in X$. A continuous linear map $D : A \to X^*$ is called a derivation if $D(ab) = D(a)\cdot b + a\cdot D(b)$, for every $a, b \in A$, and it is called an inner derivation if there exists a $\psi \in X^*$ such that $D(a) = a\cdot\psi - \psi\cdot a$, for every $a \in A$. Here "$\cdot$" denotes the module multiplications. A Banach algebra $A$ is called ($\varphi$-left) amenable if for every Banach ($\varphi$-left) $A$-bimodule $X$, every derivation $D : A \to X^*$ is inner. In the case that $X$ is a Banach $\varphi$-$A$-bimodule, i.e. $a\cdot x = x\cdot a = \varphi(a)x$, for all $a \in A$ and $x \in X$, the map $D$ has the form $D(ab) = \varphi(a)D(b) + \varphi(b)D(a)$, for all $a, b \in A$, which is called a $\varphi$-derivation on $A$.
The author had already obtained the proof of the following theorem for commutative Banach algebras. After writing this paper, Professor E. Kaniuth kindly brought the preprint [18] to my attention, in which the theorem has been similarly and simultaneously proved in the general case. So, the proof of the following theorem is referred to [18].

**Theorem 3.4.** [18] Let $A$ be a Banach algebra and $\varphi \in Hom(A,\mathbb{C})$. Then $A$ is $\varphi$-amenable if and only if $A$ is $\varphi$-left amenable. Further, $A$ has a bounded right approximate identity and is $\varphi$-amenable if and only if the maximal ideal $I_\varphi(A)$ has a bounded right approximate identity.

It is of vital importance to observe that in Theorem 3.4 the boundedness of approximate identities cannot be omitted either in $A$ or in $I_\varphi(A)$; see Section 5 (iii).

**Corollary 3.5.** By Theorems 3.4 we see that if $I_\varphi(A)$ has a bounded right approximate identity, then every $X^*$-valued continuous derivation on $A$ is zero whenever $X$ is a Banach $\varphi$-$A$-bimodule.

In various references, for particular cases, it is already known that in the spectral sets every bounded point derivation is zero, but we have not seen it in general form. It seems worthwhile to give a complete proof of it here.

**Proposition 3.6.** Let $A$ be a commutative, semisimple and regular Banach algebra and $\varphi \in \Phi(A)$. If $\{\varphi\}$ is a spectral set, then every derivation from $A$ into $X^*$ is zero whenever $X$ is a Banach $\varphi$-$A$-bimodule.

*Proof.* Let $X$ be a Banach $\varphi$-$A$-bimodule and $D : A \to X^*$ a derivation. For $x$, a fixed element of $X$, define $D_x : A \to \mathbb{C}$ by $D_x(a) := D(a)(x)$. Then $D_x$ is a $\varphi$-derivation on $A$. The set $J := \{a \in A : \varphi(a) = D_x(a) = 0\}$ is an ideal in $A$ with $Co(J) = \{\varphi\}$, where $Co(J) := \{\phi \in \Phi(A) : \phi(a) = 0, \forall a \in J\}$. To see this, let $a \in A$ and $b \in J$. Since $D_x(ab) = \varphi(a)D_x(b) + \varphi(b)D_x(a) = 0$, we have $ab \in J$. If $b \in I_\varphi(A)$, then $D_x(b^2) = 2\varphi(b)D_x(b) = 0$, which implies that $b^2 \in J$. For every $\phi \in Co(J)$ we have $\phi(b)^2 = \phi(b^2) = 0$, hence $I_\varphi(A) \subseteq I_\phi(A)$, and thus $\varphi = \phi$; thence $J = I_\varphi(A)$, since $\{\varphi\}$ is a spectral set for $A$. Let $a \in A$ with $\varphi(a) = 1$. Evidently $a^2 - a \in I_\varphi(A)$, and since $D_x|_{I_\varphi(A)} = 0$, $D_x(a^2) = D_x(a)$, so $D_x(a) = 0$ which completes the proof. □

**Remark 3.7.** Observe that in general a Banach algebra $A$ is not necessarily $\varphi$-amenable if $\{\varphi\}$ is a spectral set for $A$. For example, let $A$ be the hypergroup algebra of the Bessel-Kingman hypergroup of order $\nu = 0$. Then $A$ is $\varphi$-amenable if and only if $\varphi$ is associated with the identity character on the hypergroup [2] although every character of $A$ is a spectral set [12].

**Remark 3.8.** One may pose the question of whether $A$ is amenable if $A$ is $\varphi$-amenable for every $\varphi \in Hom(A,\mathbb{C})$ and if functionals $m_\varphi$ associated to the characters $\varphi$ are uniformly bounded. The following example from [3] shows that, in general, these assumptions cannot imply even weak amenability of $A$.

**Example:**

Let $AC$ be the set of absolutely continuous functions on the unit circle $\mathbb{T}$. For $f \in AC$, set $\|f\| = |f|_{\mathbb{T}} + \int_0^{2\pi} \left|f'(e^{i\theta})\right| d\theta$, where $|\cdot|_{\mathbb{T}}$ denotes the uniform norm on $\mathbb{T}$. Then $(AC, \|\cdot\|)$ is a Banach function algebra on $\mathbb{T}$. The Banach algebra $L^1(\mathbb{T})$ is a commutative Banach $AC$-bimodule with respect to the pointwise product of functions, and the map $f \mapsto f'$, $AC \to L^1(\mathbb{T})$, is a nonzero derivation. Thus $AC$ is not weakly amenable [3]. Let $M$ be a maximal ideal of $AC$, so that $M = \{f \in AC : f(z_0) = 0\}$ for some $z_0 \in \mathbb{T}$. Set $e_n(z) = \min\{n|z - z_0|, 1\}$. Then it is easy to check that $\{e_n\}$ is a bounded approximate identity in $M$, hence, by Corollary 3.5, there are no nonzero $\varphi_{z_0}$-derivations on $AC$. Let $m_{z_0}$ be an accumulation point of $\{e_n\}_n$ in the bidual space $AC^{**}$. Let $u \in AC$ such that $u(z_0) = 1$. Then the functional $M_{z_0} = u - m_{z_0}u$ is a $\varphi_{z_0}$-mean on $AC^*$, and since $|e_n(z)| \leq 1$, obviously $\|m_{z_0}\| \leq 2$ when $z_0$ varies on $\mathbb{T}$.

In the remaining of this section we investigate the $\varphi$-amenable Banach algebras with unique $\varphi$-means. To continue we may require the following simple lemma.

**Lemma 3.9.** Let $X$ be a linear, closed, $A$-left invariant subspace of $A^*$. If $X$ is $\varphi$-left amenable with a $\varphi$-mean $m_\varphi$, then $m_\varphi(\phi) = \delta_\varphi(\phi)$ for every $\phi \in Hom(A,\mathbb{C})$.

*Proof.* For every $\phi \in Hom(A,\mathbb{C})$ and $a \in A$ we have

$$\phi(a)m_\varphi(\phi) = m_\varphi(\phi \cdot a) = \varphi(a)m_\varphi(\phi),$$

which implies that $m_\varphi(\phi) = \delta_\varphi(\phi)$. ☐

Let $X$ be a linear, closed, $A$-left invariant subspace of $A^*$ such that $X^* \cdot X \subseteq X$. Moreover, suppose that $X^*$ with the following product is a Banach algebra,

$$\langle m \cdot n, f \rangle = \langle m, n \cdot f \rangle, \qquad \text{for every } m,n \in X^* \text{ and } f \in X. \tag{1}$$

The topological centre of $X^*$ is defined as follows:

$$Z(X^*) := \Big\{ m \in X^* : \ \text{the map } n \longrightarrow m \cdot n \ \text{ is } w^*\text{-}w^* \text{ continuous on } X^* \Big\}. \tag{2}$$

Note that $Z(X^*)$ is a closed subalgebra of $X^*$.

**Theorem 3.10.** Suppose $X \subseteq A^*$ is as above, $Hom(A,\mathbb{C}) \subseteq X$, and $\varphi \in Hom(A,\mathbb{C})$. If $X$ is $\varphi$-left amenable with a unique $\varphi$-mean $m_\varphi$, then $m_\varphi \in Z(X^*)$.

*Proof.* Let $n \in X^*$, $f \in X$, and $a \in A$. Since $X \cdot A \subseteq X$ and $X^* \cdot X \subseteq X$, we have

$$\begin{aligned} \langle m_\varphi \cdot n, f \cdot a \rangle &= \langle m_\varphi, n \cdot (f \cdot a) \rangle \\ &= \langle m_\varphi, (n \cdot f) \cdot a \rangle \\ &= \varphi(a) \langle m_\varphi, n \cdot f \rangle \\ &= \varphi(a) \langle m_\varphi \cdot n, f \rangle. \end{aligned}$$

Since $m_\varphi$ is a unique $\varphi$-mean on $X$, $\langle m_\varphi, \varphi \rangle = 1$ and $\langle n \cdot \varphi, a \rangle = \langle n, \varphi \cdot a \rangle = \varphi(a)\langle n, \varphi \rangle$, we have $m_\varphi \cdot n = \lambda_n n$, where $\lambda_n = \langle n, \varphi \rangle$. Let $n \in X^*$ and $\{n_i\}_i$ be a net in $X^*$ such that $n_i \xrightarrow{w^*} n$. Then

$$\lambda_{n_i} = \langle n_i, \varphi \rangle \to \langle n, \varphi \rangle = \lambda_n, \qquad \text{as } i \to \infty,$$

hence for every $f \in A^*$ we have

$$\begin{aligned} \langle m_\varphi \cdot n_i, f \rangle &= \langle \lambda_i m_\varphi, f \rangle \\ &= \lambda_i \cdot \langle m_\varphi, f \rangle \to \lambda_n . \langle m_\varphi, f \rangle = \langle m_\varphi \cdot n, f \rangle, \qquad \text{as } i \to \infty. \end{aligned}$$

Thus, the map $n \to m_\varphi \cdot n$ is $w^*$-$w^*$ continuous on $X^*$, hence $m_\varphi \in Z(X^*)$. ☐

**Remark 3.11.**

(i) In the above theorem, suppose that $A$ has a bounded approximate identity, bounded by 1, $X = A^*A$, and $AZ(A^{**}) \subseteq A$. We note that $X^*$ with the product defined in (1) is a Banach algebra, and $Z(X^*)$ can be identified with the right multiplier algebra of $A$, in particular $AZ(X^*) \subseteq A$; see [19, p.1295]. So, if $X$ is $\varphi$-amenable with a unique $\varphi$-mean $m_\varphi$, then the map $T_{m_\varphi} : A \to A$ defined by $T_{m_\varphi}(a) = a \cdot m_\varphi$ is a right multiplier on $A$, i.e. $T_{m_\varphi}(ab) = aT_{m_\varphi}(b)$, for every $a, b \in A$. In addition, if $A$ is semisimple and commutative, then there exits a $\psi \in C^b(\Phi(A))$ such that $\widehat{T_{m_\varphi}a} = \psi\widehat{a}$ with $\|\psi\|_\infty \leq \|m_\varphi\|$, where $\widehat{a}$ denotes the Gelfand transform of $a$; see [22, 1.2.2]. Consequently, by Lemma 3.9 we have $\psi(\phi) = \delta_\varphi(\phi)$ which implies that the character $\varphi$ is isolated in $\Phi(A)$.

(ii) It is crucial to note that the $\varphi$-left amenability of $X$ with a unique $\varphi$-mean depends on subspaces $X$ and the characters $\varphi$. For instance, if $G$ is a locally compact group, then $L^\infty(G)$ is 1-amenable with a unique 1-mean if and only if $G$ is compact; see [21]. However, subalgebras $AP(G)$ and $WAP(G)$ of $L^\infty(G)$ are always 1-amenable with the unique 1-mean [21, p.88].

(iii) Let $A$ be $L^1(K)$ when $K$ is the little q-Legendre polynomial hypergroup. Then $A$ is $\varphi$-amenable at every character $\varphi \in \Phi(A)$; and, the cardinality of $\varphi$-means is one and infinity if $\varphi$ is associated with nontrivial and trivial characters of the hypergroup, respectively [2]. However, $WAP(K)$ has only the unique $\varphi$-means at every character $\varphi$, [2, 24, 25].
We may also observe that $L^\infty(K)^*$ ($K$ is a commutative hypergroup) is isometrically isomorphic to the Banach algebra of finitely additive measures on $K$; see [14]. So, if $L^\infty(K)$ is $\varphi$-amenable with the unique $\varphi$-mean $m_\varphi$, then by part (i) the finitely additive measure associated with $m_\varphi$ belongs to the multiplier algebra of $L^1(K)$, i.e. $M(K)$ [4].

# 4 $\varphi$-Amenability of Direct Sum of Banach Algebras

In this section we study the $\varphi$-amenability of the Banach algebra $A \oplus_\phi B$, the direct sum of two given Banach algebras $A$ and $B$ where its algebraic structure depends on $\phi \in Hom(B, \mathbb{C})$. Such algebras in special cases have been studied in [1, 17].

**Definition 4.1.** Let $A$ and $B$ be Banach algebras and $\phi \in Hom(B, \mathbb{C})$. Define direct sum of $A$ and $B$, denoted by $A \oplus_\phi B$, to be the algebra over the complex numbers consisting of all ordered pairs $(a, b), a \in A, b \in B$ with coordinatewise addition and scalar multiplication, and the product of two elements $a = (a_1, a_2)$ and $b = (b_1, b_2)$ is defined by

$$a \cdot_\phi b = (a_1 b_1 + \phi(a_2) b_1 + \phi(b_2) a_1, a_2 b_2). \tag{3}$$

For every $a=(a_1,a_2)$ as above define $\|(a_1,a_2)\|_{\oplus_\phi}=\|a_1\|+\|a_2\|$. Plainly we have $\|a\cdot_\phi b\|_{\oplus_\phi}\leq \|a\|_{\oplus_\phi}\|b\|_{\oplus_\phi}$, and so $\|\cdot\|_{\oplus_\phi}$ is a norm on $A\oplus_\phi B$. If $A$ and $B$ are Banach $*$-algebras, then $(A\oplus_\phi B,\|\cdot\|_{\oplus_\phi})$ with the the involution defined by $a^*:=(a_1^*,a_2^*)$ and the algebraic product in (3) is a Banach $*$ - algebra. Obviously $(A,\|\cdot\|)$ and $(B,\|\cdot\|)$ are subalgebras of $A\oplus_\phi B$. The following proposition is the special case of [20, 1.10.13].

**Proposition 4.2.** Let $A$ and $B$ be Banach algebras. Then $\left(A\oplus_\phi B\right)^*$ is isometrically isomorphic to $A^*\times B^*$ with the norm $\|(f,g)\|=\max\{\|f\|,\|g\|\}$, $f\in A^*$ and $g\in B^*$. The isomorphism

$$\eta: A^*\times B^*\longrightarrow \left(A\oplus_\phi B\right)^*$$

is given by

$$\langle \eta(f,g),(a,b)\rangle = f(a)+g(b),$$

for every $(f,g)\in A^*\times B^*$ and $(a,b)\in A\oplus_\phi B$.

If $A^{**}$ and $B^{**}$ are equipped with the Arens product, then $A^{**}\oplus_\phi B^{**}$ is an algebra as above and the product of two elements $(m_1,m_2)$ and $(m_1',m_2')$ of $A^{**}\oplus_\phi B^{**}$ is given by

$$(m_1,m_2)\odot_\phi (m_1',m_2') = (m_1\cdot m_1'+m_2(\phi)m_1'+m_2'(\phi)m_1, m_2\cdot m_2').$$

**Proposition 4.3.** Let $A$ and $B$ be Banach algebras. Then $A^{**}\oplus_\phi B^{**}$ is isometric and algebraically isomorphic to the bidual algebra $\left(A\oplus_\phi B\right)^{**}$.

*Proof.* Define the map $J: A^{**}\oplus_\phi B^{**}\longrightarrow \left(A\oplus_\phi B\right)^{**}$ by $(m_1,m_2)\mapsto J(m_1,m_2)$ such that

$$\langle J(m_1,m_2),(f,g)\rangle = m_1(f)+m_2(g),$$

for every $(m_1,m_2)\in A^{**}\times B^{**}$ and $(f,g)\in \left(A\oplus_\phi B\right)^*$. The map $J$ is a linear isomorphism [20, 1.10.13], so we shall now check that the isomorphism is algebraic as well. For every $(f,g)\in (A\oplus B)^*$ it is easily to verify that

$$J(m_1,m_2)\cdot(f,g) = (m_1\cdot f+m_2(\phi)f, m_1(f)\phi+m_2\cdot g).$$

Let $(m_1,m_2)$, $(m_1',m_2')\in A^{**}\times B^{**}$. Then

$$\begin{aligned}
\langle J(m_1,m_2)\cdot J(m_1',m_2'),(f,g)\rangle &= \langle J(m_1,m_2), J(m_1',m_2')\cdot(f,g)\rangle \\
&= \langle J(m_1,m_2),(m_1'\cdot f+m_2'(\phi)f, m_1'(f)\phi+m_2'\cdot g)\rangle \\
&= \langle m_1, m_1'\cdot f+m_2'(\phi)f\rangle + \langle m_2, m_1'(f)\phi+m_2'\cdot g\rangle \\
&= \langle m_1,m_1'\cdot f\rangle + m_2'(\phi)m_1(f)+m_1'(f)m_2(\phi)+\langle m_2,m_2'\cdot g\rangle \\
&= \langle \left(m_1\cdot m_1'+m_2'(\phi)m_1+m_2(\phi)m_1', m_2\cdot m_2'\right),(f,g)\rangle \\
&= \langle J((m_1,m_2)\odot_\phi (m_1',m_2')),(f,g)\rangle,
\end{aligned}$$

hence

$$J\left((m_1,m_2)\odot_\phi (m'_1,m'_2)\right)=J(m_1,m_2)\cdot J(m'_1,m'_2).$$

□

**Theorem 4.4.** Let $A$ and $B$ be Banach algebras. Then the following algebraic isomorphism holds

$$Z\left((A\oplus_\phi B)^{**}\right)\cong Z(A^{**})\oplus_\phi Z(B^{**}).$$

*Proof.* To show " $\subseteq$ ", by previous proposition, we may let $(m_1,m_2)\in Z\left(A^{**}\oplus_\phi B^{**}\right)$. The map

$$\theta : A^{**}\oplus_\phi B^{**}\longrightarrow A^{**}\oplus_\phi B^{**},$$

defined by

$$(m'_1,m'_2)\mapsto (m_1,m_2)\odot_\phi (m'_1,m'_2),$$

is $w^*$ - $w^*$ continuous. Let now $(m'_i,0)\xrightarrow{w^*}(m'_1,0)$, as $i\to\infty$. Then

$$\begin{aligned}
&|\langle (m_1,m_2)\oplus_\phi (m'_i,0),(f,g)\rangle-\langle (m_1,m_2)\oplus_\phi (m'_1,0),(f,g)\rangle|\\
&\qquad=|\langle \left(m_1\cdot m'_i+m_2(\phi)m'_i,0\right),(f,g)\rangle-\langle \left(m_1\cdot m'_1+m_2(\phi)m'_1,0\right),(f,g)\rangle|\\
&\qquad=|\langle m_1\cdot m'_i,f\rangle-\langle m_1\cdot m'_1,f\rangle+m_2(\phi)\left(\langle m'_i,f\rangle-\langle m'_1,f\rangle\right)|,
\end{aligned}\tag{4}$$

and

$$\begin{aligned}
\left|\langle m_1\cdot m'_i,f\rangle-\langle m_1\cdot m'_1,f\rangle\right|\le&\left|\langle m_1\cdot m'_i,f\rangle-\langle m_1\cdot m'_1,f\rangle+m_2(\phi)\left(\langle m'_i,f\rangle-\langle m'_1,f\rangle\right)\right|\\
&+|m_2(\phi)|\left|\langle m'_i,f\rangle-\langle m'_1,f\rangle\right|.
\end{aligned}\tag{5}$$

Since $m'_i\xrightarrow{w^*}m'_1$ (as $i\to\infty$) and the inequality (4) holds, the inequality (5) implies that the map $m'_1\mapsto m_1\cdot m'_1$ is $w^*$-$w^*$ continuous on $A^{**}$. Likewise one can verify that the map $m'_2\mapsto m_2\cdot m'_2$ is also $w^*$-$w^*$ continuous on $B^{**}$, hence $(m_1,m_2)\in Z(A^{**})\oplus_\phi Z(B^{**})$.
To show " $\supseteq$ ", let $(m_1,m_2)\in Z(A^{**})\oplus_\phi Z(B^{**})$ and consider the map

$$(m'_1,m'_2)\mapsto (m_1,m_2)\odot_\phi (m'_1,m'_2)\tag{6}$$

on $A^{**}\oplus_\phi B^{**}$. Assuming $(m'_i,m''_i)\xrightarrow{w^*}(m_1,m_2)$, as $i\to\infty$, and $(f,g)\in A^*\times B^*$ imply that

$$\begin{aligned}
&|\langle (m_1,m_2)\odot_\phi (m'_i,m''_i)-(m_1,m_2)\odot_\phi (m'_1,m'_2),(f,g)\rangle|\le\\
&\qquad\left|\langle m_1\cdot m'_i-m_1\cdot m'_1,f\rangle\right|+|m_2(\phi)|\left|\langle m'_i-m'_1,f\rangle\right|\\
&\qquad+|m_1(f)|\left|\langle m''_i-m'_2,\phi\rangle\right|+\left|\langle m_2\cdot m''_i-m_2\cdot m'_2,g\rangle\right|.
\end{aligned}\tag{7}$$

Since the maps $m'_1\mapsto m_1\cdot m'_1$ and $m'_2\mapsto m_2\cdot m'_2$ are $w^*$-$w^*$ continuous on $A^{**}$ and $B^{**}$ respectively, the inequality (7) implies that the map (6) is $w^*$-$w^*$ continuous on $A^{**}\oplus_\phi B^{**}$. Thence, $(m_1,m_2)\in Z\left(A^{**}\oplus_\phi B^{**}\right)$, and as a result the identity map is an algebraic isomorphism. □

**Proposition 4.5.** Let $A$ and $B$ be Banach algebras. Then $Hom(A \oplus_\phi B, \mathbb{C})$ is isomorphic to $Hom(A, \mathbb{C}) \times \{\phi\}$.

*Proof.* Define $\rho : Hom(A,\mathbb{C}) \times \{\phi\} \to Hom(A \oplus_\phi B, \mathbb{C})$ by $(\psi, \phi) \mapsto \rho_{(\psi,\phi)}$, where

$$\rho_{(\psi,\phi)}(a_1, a_2) := \psi(a_1) + \phi(a_2), \qquad (a_1, a_2) \in A \oplus_\phi B.$$

The map $\rho_{(\psi,\phi)}$ is clearly linear and bounded, and also multiplicative. To verify the latter, let $a = (a_1, a_2)$ and $b = (b_1, b_2)$ be in $A \oplus_\phi B$. Then

$$\begin{aligned}
\rho_{(\psi,\phi)}(a \cdot_\phi b) &= \rho_{(\psi,\phi)}(a_1b_1 + \phi(a_2)b_1 + \phi(b_2)a_1, a_2b_2) \\
&= \psi(a_1b_1) + \phi(a_2)\psi(b_1) + \phi(b_2)\psi(a_1) + \phi(a_2b_2) \\
&= \psi(a_1)\psi(b_1) + \phi(a_2)\psi(b_1) + \phi(b_2)\psi(a_1) + \phi(a_2)\phi(b_2) \\
&= \rho_{(\psi,\phi)}(a) \cdot \rho_{(\psi,\phi)}(b).
\end{aligned}$$

In order to prove the converse of the theorem, let $\theta \in Hom(A \oplus_\phi B, \mathbb{C})$. Obviously $\theta|_B = \phi'$ for some $\phi' \in Hom(B, \mathbb{C})$ and there exists a $\psi \in Hom(A, \mathbb{C})$ such that $\theta\ |_A = \psi$; thus, $\theta(a_1, a_2) = \psi(a_1) + \phi'(a_2)$. Let $a_1 \in A$ such that $\psi(a_1) \neq 0$. Then $\theta((a_1, a_2) \cdot (0, b_2)) = \phi(b_2)\psi(a_1) + \phi'(a_2)\phi'(b_2)$, on the other hand, since $\theta$ is multiplicative, we have $\theta((a_1, a_2) \cdot (0, b_2)) = \psi(a_1)\phi'(b_2) + \phi'(a_2)\phi'(b_2)$, which implies that $\phi(b_2) = \phi'(b_2)$, for every $b_2 \in B$; thence $\theta = \rho_{(\psi,\phi)}$. $\square$

**Remark 4.6.** Let $\{e_i\}_i$ be a bounded left approximate identity for $B$ such that $\lim\limits_{i\to\infty} \phi(e_i) = 1$. For every $(a,b) \in A \oplus_\phi B$ we have

$$\|(a,b) - (0, e_i) \cdot (a,b)\|_{\oplus_\phi} = \|\,(\phi(e_i) - 1)\,a\| + \|e_ib - b\|,$$

hence $\{(0, e_i)\}$ is a bounded left approximate identity for $A \oplus_\phi B$.

**Theorem 4.7.** Let $A$ and $B$ be Banach algebras and $\varphi \in Hom(A, \mathbb{C})$. Then $A \oplus_\phi B$ is $\rho_{(\varphi,\phi)}$-amenable if and only if $A$ is $\varphi$-amenable.

*Proof.* Let $A$ be $\varphi$-amenable. Then there exists a $m_\varphi \in A^{**}$ such that $m_\varphi(\varphi) = 1$ and $m_\varphi(f \cdot a) = \varphi(a)m_\varphi(f)$ for every $f \in A^*$ and $a \in A$. For any pairs $(f, g) \in A^* \times B^*$ and $(a,b) \in A \oplus_\phi B$, $(f,g) \cdot (a,b) = (f \cdot a + \phi(b)f, f(a)\phi + g \cdot b)$, and we suppose $\rho_{\varphi,\phi}$ is as in the preceding proposition. Then

$$\begin{aligned}
\langle (m_\varphi, 0), (f,g) \cdot (a,b)\rangle &= \langle (m_\varphi, 0), (f \cdot a + \phi(b)f, f(a)\phi + g \cdot b)\rangle \\
&= \langle m_\varphi, f \cdot a + \phi(b)f\rangle \\
&= \varphi(a)\langle m_\varphi, f\rangle + \phi(b)\langle m_\varphi, f\rangle \\
&= \rho_{(\varphi,\phi)}(a,b)\, m_\varphi(f) \\
&= \rho_{(\varphi,\phi)}(a,b)\langle (m_\varphi, 0), (f,g)\rangle,
\end{aligned}$$

and $\langle (m_\varphi, 0), \rho_{\varphi,\phi}\rangle = 1$; hence, $A \oplus_\phi B$ is $\rho_{(\varphi,\phi)}$-amenable.
To show the converse, assume $(m'_\varphi, m'_\phi)$ is a $\rho_{\varphi,\phi}$-mean on $\left(A \oplus_\phi B\right)^*$. We shall check that $\langle m'_\varphi, \varphi\rangle \neq 0$. Since $(m'_\varphi, m'_\phi)$ is a $\rho_{\varphi,\phi}$-mean,

$$\langle (m'_\varphi, m'_\phi), (f, 0) \cdot (a, 0)\rangle = \varphi(a) m'_\varphi(f),$$

hence

$$m'_\varphi(f \cdot a) + f(a) m'_\phi(\phi) = \varphi(a) m'_\varphi(f). \tag{8}$$

Let $a \in A$ with $\varphi(a) \neq 0$. For $f = \varphi$ the equality (8) yields $m'_\phi(\phi) = 0$. As a result, $m'_\varphi(f \cdot a) = \varphi(a) m'_\varphi(f)$, for every $f \in A^*$ and $a \in A$. Since $(m'_\varphi, m'_\phi)(\rho_{\varphi,\phi}) = 1$, we may have $m'_\varphi(\varphi) = 1$ which completes the proof. $\square$

We note that, according to the previous theorem, the character amenability of $A \oplus_\phi B$ is independent of any amenability condition on $B$.

# 5 Examples

Here are examples of Banach function algebras for which their $\varphi$-amenability relies on their characters.

(i) Let $A$ be a unital uniform algebra on a nonempty compact space $\Omega$. Suppose that $K_A = \{\lambda \in A' : \|\lambda\| = \langle 1, \lambda\rangle = 1\}$ and $\Gamma_0(A)$ is the set of extremal points of $K_A$. Then [7, 4.3.5] and Theorem 3.4 show that $x \in \Gamma_0(A)$ if and only if $A$ is $\varphi_x$-amenable, where $\varphi_x \in \Phi(A) \cong \Omega$.

(ii) Let $A(\overline{\mathbb{D}}) = \left\{ f \in C(\overline{\mathbb{D}}) : f|_{\mathbb{D}} \text{ is analytic}\right\}$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the open unit disc. Then $(A(\overline{\mathbb{D}}), |\cdot|_{\overline{\mathbb{D}}})$ is a commutative Banach algebra where $|f|_{\overline{\mathbb{D}}} = \sup\{|f(z)| : z \in \overline{\mathbb{D}}\}$, which is called the disc algebra. Since $\Gamma_0(A(\overline{\mathbb{D}})) = \mathbb{T}$, by part (i), $A(\overline{\mathbb{D}})$ is $\varphi_z$-amenable if and only if $z \in \mathbb{T}$. Observe that for every $z \in \mathbb{D}$ there exists a nonzero continuous point derivation at $z$ on $A(\overline{\mathbb{D}})$ of the form $f \mapsto \alpha f'(z)$ [7, 4.3.13], for some $\alpha \in \mathbb{C}$; hence, $A(\overline{\mathbb{D}})$ is not weakly amenable [3, 1.5].

(iii) Let $K$ be a commutative compact hypergroup. Then $(C(K), \|\cdot\|_\infty)$ is a proper Segal algebra in $L^1(K)$. Since every maximal ideal in $L^1(K)$ has a bounded approximate identity, [1], the maximal ideals in $C(K)$ have only unbounded approximate identities [5]. However, the functional $m_\varphi := \pi(\varphi)/\|\varphi\|_2^2$ is the unique $\varphi$-mean on $C(K)^*$, for every $\varphi \in \Phi(C(K))$.

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